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Non-commutative ADE geometries as holomorphic wave equations

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Abstract

Borrowing ideas from the relation between classical and quantum mechanics, we study a non-commutative elevation of the ADE geometries involved in building Calabi-Yau manifolds. We derive the corresponding geometric hamiltonians and the holomorphic wave equations representing these non-commutative geometries. The spectrum of the holomorphic waves is interpreted as the quantum moduli space. Quantum A_1 geometry is analyzed in some details and is found to be linked to the Whittaker differential equation.

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1 Introduction

In recent years, there has been a great deal of interest in non-commutative (NC) spaces in connection with string theory. Common to many of these studies is that the non commutativity stems from the D-brane physics in the presence of a B-field [1]. Similar NC structures have been applied to Calabi-Yau compactifications. The underlying idea in this context is to express the non commutativity in terms of discrete isometries of orbifolds. This was successfully done in [2] for the quintic, and it has been extended to K3 surfaces [3, 4] and higher-dimensional orbifolds [5]. The NC aspect of such hypersurfaces is also important for the stringy resolution of singularities as it offers an alternative to the standard resolutions obtained by deformations of the complex or Kähler structures of the Calabi-Yau manifolds.

An objective of the present paper is to develop a new and essentially non-geometric approach to NC Calabi-Yau manifolds, based on ideas from quantum mechanics. This also offers a new take on the moduli space of resolved singularities. In this paper, a moduli space is meant to denote a space spanned by the degrees of freedom in the system. Our focus is on ADE geometries, which we represent by certain holomorphic, partial differential operators. Such an operator acts on the space of holomorphic functions Ψ on $\mathbb{C}^3$, thereby defining a wave equation. The spectrum of wave functions solving this equation is accordingly interpreted as the moduli space of the NC elevation of the associated ADE geometry. We consider in some details the case A_1, and we find that it is linked to the Whittaker differential equation.

Our wave-functional approach may be applied to more general geometries than the ADE spaces. It thus offers a whole new description of NC elevations of ordinary geometries. A more general exposition may be found in [6] while a more conventional approach to NC Calabi-Yau manifolds may be found in [7, 8].

The present paper is organized as follows: In Section 2, we outline how our NC elevations of ordinary geometries mimic the quantization of classical mechanics in the hamiltonian formalism. Section 3 concerns the quantization of the ADE geometries, while the wave-equation representations of the resulting NC geometries are discussed in Section 4. Section 5 contains some concluding remarks.

2 Basic correspondence

Our construction of NC ADE geometries as elevations of ordinary, commutative ADE geometries is based on an extension of the relation between classical and quantum mechanics. We are thus exploring a similarity between commutative ADE geometries and classical mechanics on one hand, and NC ADE geometries and quantum mechanics on the other hand. The basic ideas are outlined in the following.

2.1 Ordinary ADE geometries and classical mechanics

A hypersurface in the three-dimensional complex space $\mathbb{C}^3$ generated by $(z^1, z^2, z^3) = (x, y, z)$ may be described by an algebraic equation of the form

$$V(x, y, z) = \epsilon. \tag{2.1}$$

The explicit (polynomial) potentials $V(x, y, z)$ of our interest will be specified below. The parameter ϵ is independent of the complex coordinates (x, y, z), and may be seen as parameterizing the family or orbit of hypersurfaces characterized by a given potential V. It is observed that a point $(x, y, z) \in \mathbb{C}^3$ can lie on at most one hypersurface in a given orbit. Since ϵ is constant, a point (x_0, y_0, z_0) on the hypersurface (2.1) is a singular point if the gradient $\nabla V = (\partial_x V, \partial_y V, \partial_z V)$ vanishes at that point:

$$\nabla V(x_0, y_0, z_0) = (0, 0, 0). \tag{2.2}$$

It should be evident that the above extends to hypersurfaces in the higher-dimensional complex spaces $\mathbb{C}^n$. The ADE geometries of our interest are all singular at exactly one point. The associated potentials will be chosen such that the ADE geometries correspond to $\epsilon = 0$ in their respective orbits, and such that the singularities appear at the origin $(x_0, y_0, z_0) = (0, 0, 0)$.

The hamiltonian description of classical mechanics is quite analogous. One may be interested in characterizing the configurations corresponding to a fixed energy E. This amounts to solving the equation

$$\mathcal{H}(q_1, \ldots, q_n, p_1, \ldots, p_n) = E \tag{2.3}$$

where $\mathcal{H}$ denotes the hamiltonian. The solutions define a hypersurface in the $2n$-dimensional phase space. Up to some well-known signs, Hamilton's equations express the time derivatives of the canonical coordinates q_j, p_j, $j = 1, \ldots, n$, in terms of the gradient of the hamiltonian. A singular point of the fixed-energy hypersurface thus corresponds to the simultaneous vanishing of all these time derivatives.

With this analogy we are thus considering a similarity between the orbit of hypersurfaces based on V and the physical system described by $\mathcal{H}$. The individual hypersurfaces characterized by ϵ play the roles of specific energy levels given by E. Singularities appear where the associated gradients vanish.

There are of course important differences and further similarities between these two scenarios. Here, though, we will not be concerned with them. Rather, our objective is to explore the consequences of mimicking the quantization of classical mechanics in the realm of hypersurfaces of the form (2.1).

2.2 NC ADE geometries and quantum mechanics

We will consider a combination of the Heisenberg and Schrödinger pictures of quantization. As part of our construction we thus replace the complex coordinates z^j by holomorphic operators Z^j in analogy with the promotion of the canonical variables to quantum operators. We also borrow the idea of promoting the classical hamiltonian to a differential operator acting on a space of wave functions where the eigenfunctions correspond to the stationary states, whereas the eigenvalues represent the allowed energy levels. In our case, the potential V is replaced by a differential operator whose eigenfunctions will be certain holomorphic functions, while the eigenvalues will label the NC geometries we may represent in this picture.

The geometric analogue of the Schrödinger wave equation which we will discuss reads

$$V(X, Y, Z)\Psi = \epsilon\Psi. \tag{2.4}$$

We naturally require that this reduces to (2.1) in the classical (commutative) limit. We may decompose the 'quantum potential' $V(X, Y, Z)$ as

$$V(X, Y, Z) = H + V(x, y, z) \tag{2.5}$$

where the holomorphic (partial) differential operator H vanishes in the classical limit:

$$H \to 0. \tag{2.6}$$

The 'geometric hamiltonian' H is constructed by replacing the coordinates z^j by a differential-operator realization of the NC coordinates Z^j subject to a normal-ordering procedure to be discussed below. This also justifies the use of the same symbol V to denote the quantum potential as in the classical case. Keeping the decomposition (2.5) and the classical limit (2.6) in mind, the partial differential operators Z^j may be viewed as NC perturbations of the original commutative coordinates z^j where the perturbative terms vanish in the classical limit.

We will derive and study differential equations of the form

$$H(x, y, z; \partial_x, \partial_y, \partial_z)\Psi(x, y, z) = (\epsilon - V(x, y, z))\, \Psi(x, y, z), \tag{2.7}$$

where the ADE geometries correspond to $\epsilon = 0$. Deriving these equations essentially amounts to devising an appropriate normal ordering of the NC coordinates. This is discussed in the following section. Working out the corresponding quantum moduli space amounts to finding the spectrum of eigenfunctions Ψ in (2.7). This is a highly non-trivial task as it requires solving complicated partial differential equations. The general solution is beyond the scope of the present work, though we will present the solution in the case of A_1.

In brief, our construction offers a novel and essentially non-geometric representation of NC ADE geometries as holomorphic wave equations on $\mathbb{C}^3$. The associated moduli spaces of the

NC ADE geometries are given in terms of the spectrum of holomorphic waves solving these differential equations.

It is well known that a singular ADE geometry may be deformed by adding a polynomial term $f(x,y,z)$ to the defining potential $V(x,y,z)$, where either $f(0,0,0) \neq (0,0,0)$ or $\nabla f(0,0,0) \neq 0$. The corresponding NC elevation is represented by a wave equation of the form

$$H(x,y,z;\partial_x,\partial_y,\partial_z)\Psi(x,y,z) = -(V(x,y,z)+f(x,y,z))\Psi(x,y,z). \tag{2.8}$$

NC elevations of such deformations will not be considered further here.

3 Quantization of ADE geometries

For the sake of simplicity we will limit our analysis to the complex K3 surfaces being an important example of compact Calabi-Yau manifolds. These surfaces play a crucial role in the study of type II superstrings and in the geometric engineering of quantum field theories embedded in superstring theory [9, 10, 11, 12]. A K3 surface can have singularities corresponding to contracting two-spheres. The intersection matrix of these two-spheres is then given by the Cartan matrix of a Lie algebra, and one may naturally distinguish between three types of singularities [13], namely (**a**) *ordinary singularities* classified by the ordinary ADE Lie algebras, (**b**) *affine singularities* classified by the affine $\widehat{ADE}$ Kac-Moody algebras, and (**c**) *indefinite singularities* classified by the indefinite Lie algebras.

Here we focus on ordinary ADE singularities, while a similar analysis is possible for the affine extensions. Near such an ordinary singularity, a K3 surface may be viewed as an ALE space defined by an orbifold structure $\mathbb{C}^2/G$, where G is a discrete group depending on the ADE singularity in question. These orbifolds can be expressed as hypersurfaces in $\mathbb{C}^3$ (2.1) where $\epsilon = 0$,

$$V(x,y,z) = 0, \tag{3.1}$$

and with potentials given by [14][1]

$$\begin{aligned}
A_{n-1}: &\qquad V_{A_{n-1}}(x,y,z) := x^2+y^2+z^n, \\
D_n: &\qquad V_{D_n}(x,y,z) := x^2+y^2z+z^{n-1}, \\
E_6: &\qquad V_{E_6}(x,y,z) := x^2+y^3+z^4, \\
E_7: &\qquad V_{E_7}(x,y,z) := x^2+y^3+yz^3, \\
E_8: &\qquad V_{E_8}(x,y,z) := x^2+y^3+z^5.
\end{aligned} \tag{3.2}$$

[1]We have chosen to represent the A-series by $x^2+y^2+z^n$ instead of $uv+z^n$. The two representations are related by the invertible transformation $(x,y) \to (u,v) = (x+iy, x-iy)$. As will become clear, the choice (x,y) renders the quantization straightforward.

The indices indicate the ranks of the Lie algebras. As already mentioned, these hypersurfaces have singularities at the origin of $\mathbb{C}^3$. It is also well known that the singularity of any one of these hypersurfaces may be resolved in two ways, either by deforming the complex structure of the surface (changing the shape), or by varying its Kähler structure (changing the size). Here we are not interested in such deformations or resolutions, but rather in a NC elevation of the ADE spaces defined by the polynomial constraint equations (3.1,3.2). Following the previous section, we will introduce a quantization procedure in which these NC ADE spaces are constructed by imposing polynomial constraints similar to (3.1,3.2) on a NC generalization of $\mathbb{C}^3$.

It is noted that one may also consider K3 surfaces with singularities described by the $BCFG$ Lie algebras. The potentials defining these complex surfaces are in general multiple-valued functions [11, 12], unlike the ADE cases in (3.2), and will not be discussed here.

We now turn to the construction of the NC embedding space. The ordinary n-dimensional complex space $\mathbb{C}^n$ is parameterized by the n complex variables z^j, $j = 1, \ldots, n$. We will parameterize its NC counterpart $\mathbb{C}^n_\Theta$ by Z^j, $j = 1, \ldots, n$, satisfying

$$[Z^j, Z^k] = 2\Theta^{jk}, \tag{3.3}$$

where Θ^{jk} is an anti-symmetric complex tensor. We wish to add some comments on this non commutativity. The first comment concerns the fact that all anti-symmetric matrices of odd dimension are singular, i.e., not invertible. A real hamiltonian system, on the other hand, is always even-dimensional since each position variable is accompanied by a conjugate momentum variable. The singular property of Θ thus restricts the way the operators Z^j may be expressed in terms of 'phase-space' variables, see (3.5).

The second comment concerns the complex nature of the structure constants Θ^{jk}. They can be viewed as a complexification of the real Seiberg-Witten parameters known to be related to the NS-NS B-field in the description of real Moyal space [1]. In type IIB superstring theory this complexity may have its origin in terms of a complexified Kähler form $\mathcal{J} = J_K + iB_{NS}$ or in terms of a complex combination of the two B-fields (RR and NS-NS), i.e., $B = B_{NS} + iB_R$.

The third comment is that the parameters Θ^{jk} play a role similar to (the normalized) Planck's constant $\hbar$ appearing in the Heisenberg commutation relations of non-relativistic quantum mechanics:

$$[\mathcal{P}, \mathcal{X}] = -i\hbar. \tag{3.4}$$

Here $\mathcal{X}$ and $\mathcal{P}$ are the usual position and momentum operators, respectively. As is well known, they admit a representation in which $\mathcal{X} = x$ and $\mathcal{P} = -i\hbar\partial_x$. Motivated by this, we wish to realize the NC coordinates Z^j (satisfying (3.3)) in terms of a linear combination of $2n$

'phase-space' variables, $\mathcal{Z}^j$ and $\mathcal{P}_j$, as follows

$$Z^j = \mathcal{Z}^j + \sum_{k=1}^{n} \Theta^{jk} \mathcal{P}_k \tag{3.5}$$

where $\mathcal{P}_j$ and $\mathcal{Z}^k$ are quantum operators satisfying

$$[\mathcal{P}_j, \mathcal{Z}^k] = \delta_j^k, \qquad [\mathcal{Z}^j, \mathcal{Z}^k] = [\mathcal{P}_j, \mathcal{P}_k] = 0. \tag{3.6}$$

Representing the variables $(\mathcal{Z}^j, \mathcal{P}_j)$ as $(z^j, \frac{\partial}{\partial z^j})$, we may represent the NC coordinates, Z^j, as first-order differential operators:

$$Z^j = z^j + \sum_{k=1}^{n} \Theta^{jk} \partial_k, \qquad \partial_k = \frac{\partial}{\partial z^k}. \tag{3.7}$$

In this representation, the NC coordinates are thus seen to act as non-trivial, holomorphic, partial differential operators on the space of holomorphic functions $\Psi(z_1, \ldots, z_n)$ on $\mathbb{C}^n$, and we are one step closer to the geometric analogue of the Schrödinger equation discussed above.

For invertible Θ^{jk}, in which case the dimension n must be even, one may introduce the 'gauge potential' $A_j = \sum_{k=1}^{n} \Theta_{jk} z^k$. The realization (3.7) may then be re-expressed in terms of the 'covariant' derivative $D_j = \partial_j + A_j$ as

$$D_j = \sum_{k=1}^{n} \Theta_{jk} Z^k. \tag{3.8}$$

This indicates that the NC elevation in these cases behaves as switching on an external constant magnetic field $B^i = \varepsilon^{ijk} \partial_j A_k = -\varepsilon^{ijk} \Theta_{jk}$. Since our main interest is based on $n = 3$, we will not elaborate on this observation.

Referring to the notation in (3.2), we will parameterize $\mathbb{C}^3_\Theta$ by the NC coordinates (X, Y, Z) satisfying

$$[X, Y] = 2\alpha, \qquad [Y, Z] = 2\beta, \qquad [Z, X] = 2\gamma \tag{3.9}$$

where α, β, γ are (commutative) structure constants. It is noted that this algebra is equivalent to a central extension of the direct sum of three $u(1)$s. That is, the $u(1)$s are originally generated by (the commuting variables) X, Y, Z while the central element is denoted I. To complete the interpretation of (3.9) as this central extension, the structure constants appearing on the right-hand sides of the commutators should all be multiplied by I.

Following (3.5) and (3.7), the representations of (X, Y, Z) of our interest now read

$$\begin{aligned} X &= \mathcal{X} + \alpha \mathcal{P}_y - \gamma \mathcal{P}_z = x + \alpha \partial_y - \gamma \partial_z, \\ Y &= \mathcal{Y} + \beta \mathcal{P}_z - \alpha \mathcal{P}_x = y + \beta \partial_z - \alpha \partial_x, \\ Z &= \mathcal{Z} + \gamma \mathcal{P}_x - \beta \mathcal{P}_y = z + \gamma \partial_x - \beta \partial_y. \end{aligned} \tag{3.10}$$

It is emphasized that these operators act on the local coordinates as $[X, x] = 0$, $[X, y] = \alpha$, $[X, z] = -\gamma$ etc. It is also noted that one may consider various degrees of non commutativity corresponding to

$$\begin{aligned}
\alpha &\neq 0, \qquad \beta = \gamma = 0, \qquad \text{or a cyclic permutation,} \\
\alpha\beta &\neq 0, \qquad \gamma = 0, \qquad \text{or a cyclic permutation,} \\
\alpha\beta\gamma &\neq 0.
\end{aligned} \tag{3.11}$$

The remaining case where $\alpha = \beta = \gamma = 0$ merely corresponds to classical geometry. Obviously, the possibility $\alpha\beta\gamma \neq 0$ has the highest degree of non commutativity.

Our next objective is to define the NC elevation of the potentials $V(x, y, z)$. As in other 'quantization' schemes, the naive substitution

$$(x, y, z) \to (X, Y, Z) \tag{3.12}$$

is ambiguous due to the simple fact that $xy = yx$ while $XY \neq YX$ if $\alpha \neq 0$, for example, and one is faced with an ordering problem. According to (3.2), we need to treat y^2z and yz^3, as these monomials appear in the D_n and E_7 potentials, respectively. To this end, and to put it into a more general context, we introduce the homogeneous polynomials

$$M_m(u, v) = \sum_{j=0}^{m} a_j u^j v u^{m-j}, \qquad \sum_{j=0}^{m} a_j = 1 \tag{3.13}$$

of degree $m + 1$ where m is a non-negative integer. The arguments, u and v, may be NC variables, and $M_m(u, v)$ is seen to reduce to the monomial $u^m v$ if $[u, v] = 0$.

Now, in our case we are thus interested in

$$\begin{aligned}
M_m(X, Y) &= \sum_{j=0}^{m} a_j X^j Y X^{m-j} \\
&= \sum_{j=0}^{m} a_j (\mathcal{X} + \alpha\mathcal{P}_y - \gamma\mathcal{P}_z)^j (\mathcal{Y} + \beta\mathcal{P}_z - \alpha\mathcal{P}_x)(\mathcal{X} + \alpha\mathcal{P}_y - \gamma\mathcal{P}_z)^{m-j},
\end{aligned} \tag{3.14}$$

and we find that it may be written in the following form:

Lemma

$$\begin{aligned}
M_m(X, Y) &= \mathcal{Y}X^m + X^m(\beta\mathcal{P}_z - \alpha\mathcal{P}_x) + \alpha \sum_{j=0}^{m} (2j - m) a_j X^{m-1}, \\
M_m(X, X) &= X^{m+1}.
\end{aligned} \tag{3.15}$$

Up to commutative (hence trivial) re-arrangements (within X^s), the ordering of the right-hand side has phase-space coordinates to the left of the phase-space momenta. We will refer to this ordering as *normal ordering*.

Our proposal for a 'natural' quantization procedure that elevates an ordinary hypersurface to a NC hypersurface now goes as follows. Let the classical hypersurface be defined by the vanishing of a polynomial, as in the case of the ADE manifolds (3.1,3.2). Since our prime goal is to construct NC elevations of these ADE spaces, we will restrict ourselves to the situation where each monomial summand is of the form $x^m y$ or z^s, or similar monomials in $\{x, y, z\}$ obtained by replacing x, y or z by one of the other coordinates. Each of these monomials is then replaced by the most general homogeneous polynomial in the corresponding NC coordinates $\{X, Y, Z\}$ (as in the first line of (3.14), for example) satisfying that the result of normal ordering it must itself be a homogeneous polynomial in the phase-space variables of the NC coordinates. It is of course also required that the NC polynomial is properly normalized so that it reduces to the original polynomial in the classical limit where $(X, Y, Z) \to (x, y, z)$. For the class of polynomials $M_m(X, Y)$, this means that the right-hand side of (3.15) must be homogeneous in the phase-space variables, which is ensured provided

$$\sum_{j=0}^{m}(2j - m)a_j = 0. \tag{3.16}$$

The symmetrized polynomial, where $a_0 = \ldots = a_m = \frac{1}{m+1}$, is seen to satisfy this condition, showing that a solution exists for all m. It may appear surprising, though, that the thus defined set of 'quantizations' of a given classical polynomial consists of *one* polynomial only. This follows straightforwardly, though, from the lemma with (3.16) imposed, since the first part of the right-hand side of (3.15) is *independent* of $\{a_0, \ldots, a_m\}$, and from the fact that the quantization of z^s is trivial. It also indicates that our quantization procedure for a complex hypersurface like the ADE spaces (3.1,3.2) results in a *unique* NC hypersurface. In brief, the quantization procedure replaces uniquely the classical (commutative) monomials of the form $x^m y$ or z^s by homogeneous polynomials of degree $m + 1$ or s, respectively, in the phase-space variables associated to the NC coordinates $\{X, Y, Z\}$.

Let us illustrate the uniqueness from the point of view of the relations following from the commutative nature of the structure constants (3.9). Recall that $\sum_{j=0}^{m} a_j = 1$ ensures that the NC polynomial reduces to its commutative origin in the classical limit $(X, Y, Z) \to (x, y, z)$, while $\sum_{j=0}^{m}(2j - m)a_j = 0$ ensures homogeneity of the NC counterpart of a classical monomial. For $m = 1$ there is only one solution to these two constraints: $a_0 = a_1 = 1/2$. For $m = 2$ there is the one-parameter family of solutions

$$a_0 = a, \qquad a_1 = 1 - 2a, \qquad a_2 = a, \tag{3.17}$$

but

$$ZY^2 - 2YZY + Y^2 Z = [Z, Y]Y - Y[Z, Y] = 0 \tag{3.18}$$

according to the aforementioned commutative nature of the structure constants. Likewise for $m = 3$, where

$$a_0 = b, \qquad a_1 = \frac{1}{2} - 2b + c, \qquad a_2 = \frac{1}{2} + b - 2c, \qquad a_2 = c \tag{3.19}$$

is the general solution, we have

$$YZ^3 - 2ZYZ^2 + Z^2YZ = (YZ^2 - 2ZYZ + Z^2Y)Z = 0, \tag{3.20}$$

for example. This demonstrates again that our quantization procedure results in a unique NC hypersurface which we may then choose to represent in its symmetrized form (corresponding to $a = 1/3$ for $m = 2$, and $b = c = 1/4$ for $m = 3$):

$$\begin{aligned}
A_{n-1}: &\qquad V_{A_{n-1}}(X, Y, Z) := X^2 + Y^2 + Z^n, \\
D_n: &\qquad V_{D_n}(X, Y, Z) := X^2 + \frac{1}{3}\left(ZY^2 + YZY + Y^2Z\right) + Z^{n-1}, \\
E_6: &\qquad V_{E_6}(X, Y, Z) := X^2 + Y^3 + Z^4, \\
E_7: &\qquad V_{E_7}(X, Y, Z) := X^2 + Y^3 + \frac{1}{4}\left(YZ^3 + ZYZ^2 + Z^2YZ + Z^3Y\right), \\
E_8: &\qquad V_{E_8}(X, Y, Z) := X^2 + Y^3 + Z^5.
\end{aligned} \tag{3.21}$$

Upon replacing the operators X, Y and Z by their differential-operator representations given in (3.10), it is straightforward to write down the corresponding holomorphic wave equations (2.4). This is discussed below. It is also stressed that, by construction, the NC nature of the quantized ADE spaces is inherited from the ambient space $\mathbb{C}^3_\Theta$. An immediate way of seeing this is that the non-commutativity in either case is governed by the same set of structure constants α, β, γ.

4 Wave-equation representation

Here we list the differential-operator representations of the NC ADE potentials outlined above:

$$
\begin{aligned}
V_{A_{n-1}}(X,Y,Z) &= \sum_{j=0}^{2}\sum_{k=0}^{j}\begin{pmatrix}2\\ j\end{pmatrix}\begin{pmatrix}j\\ k\end{pmatrix}(-1)^k\left(\alpha^{j-k}\gamma^k x^{2-j}\partial_y^{j-k}\partial_z^k+\beta^{j-k}\alpha^k y^{2-j}\partial_z^{j-k}\partial_x^k\right)\\
&+ \sum_{j=0}^{n}\sum_{k=0}^{j}\begin{pmatrix}n\\ j\end{pmatrix}\begin{pmatrix}j\\ k\end{pmatrix}(-1)^k\gamma^{j-k}\beta^k z^{n-j}\partial_x^{j-k}\partial_y^k,\\
V_{D_n}(X,Y,Z) &= \sum_{j=0}^{2}\sum_{k=0}^{j}\begin{pmatrix}2\\ j\end{pmatrix}\begin{pmatrix}j\\ k\end{pmatrix}(-1)^k\Big(\alpha^{j-k}\gamma^k x^{2-j}\partial_y^{j-k}\partial_z^k\\
&\quad+\beta^{j-k}\alpha^k y^{2-j}\left(z+\gamma\partial_x-\beta\partial_y\right)\partial_z^{j-k}\partial_x^k\Big)\\
&+ \sum_{j=0}^{n-1}\sum_{k=0}^{j}\begin{pmatrix}n-1\\ j\end{pmatrix}\begin{pmatrix}j\\ k\end{pmatrix}(-1)^k\gamma^{j-k}\beta^k z^{n-1-j}\partial_x^{j-k}\partial_y^k,\\
V_{E_6}(X,Y,Z) &= \sum_{j=0}^{2}\sum_{k=0}^{j}\begin{pmatrix}2\\ j\end{pmatrix}\begin{pmatrix}j\\ k\end{pmatrix}(-1)^k\alpha^{j-k}\gamma^k x^{2-j}\partial_y^{j-k}\partial_z^k\\
&+ \sum_{j=0}^{3}\sum_{k=0}^{j}\begin{pmatrix}3\\ j\end{pmatrix}\begin{pmatrix}j\\ k\end{pmatrix}(-1)^k\beta^{j-k}\alpha^k y^{3-j}\partial_z^{j-k}\partial_x^k\\
&+ \sum_{j=0}^{4}\sum_{k=0}^{j}\begin{pmatrix}4\\ j\end{pmatrix}\begin{pmatrix}j\\ k\end{pmatrix}(-1)^k\gamma^{j-k}\beta^k z^{4-j}\partial_x^{j-k}\partial_y^k,\\
V_{E_7}(X,Y,Z) &= \sum_{j=0}^{2}\sum_{k=0}^{j}\begin{pmatrix}2\\ j\end{pmatrix}\begin{pmatrix}j\\ k\end{pmatrix}(-1)^k\alpha^{j-k}\gamma^k x^{2-j}\partial_y^{j-k}\partial_z^k\\
&+ \sum_{j=0}^{3}\sum_{k=0}^{j}\begin{pmatrix}3\\ j\end{pmatrix}\begin{pmatrix}j\\ k\end{pmatrix}(-1)^k\Big(\beta^{j-k}\alpha^k y^{3-j}\partial_z^{j-k}\partial_x^k\\
&\quad+\gamma^{j-k}\beta^k z^{3-j}\left(y+\beta\partial_z-\alpha\partial_x\right)\partial_x^{j-k}\partial_y^k\Big),\\
V_{E_8}(X,Y,Z) &= \sum_{j=0}^{2}\sum_{k=0}^{j}\begin{pmatrix}2\\ j\end{pmatrix}\begin{pmatrix}j\\ k\end{pmatrix}(-1)^k\alpha^{j-k}\gamma^k x^{2-j}\partial_y^{j-k}\partial_z^k\\
&+ \sum_{j=0}^{3}\sum_{k=0}^{j}\begin{pmatrix}3\\ j\end{pmatrix}\begin{pmatrix}j\\ k\end{pmatrix}(-1)^k\beta^{j-k}\alpha^k y^{3-j}\partial_z^{j-k}\partial_x^k\\
&+ \sum_{j=0}^{5}\sum_{k=0}^{j}\begin{pmatrix}5\\ j\end{pmatrix}\begin{pmatrix}j\\ k\end{pmatrix}(-1)^k\gamma^{j-k}\beta^k z^{5-j}\partial_x^{j-k}\partial_y^k.
\end{aligned}
\tag{4.1}
$$

The associated wave equations are defined by (2.4) for $\epsilon = 0$. Below follows an analysis of the case A_1.

4.1 The case A_1

We consider

$$V_{A_1}(X,Y,Z)\Psi = \left(X^2+Y^2+Z^2\right)\Psi = \epsilon\Psi \tag{4.2}$$

where $\epsilon = 0$ corresponds to the NC A_1 geometry. In terms of differential operators we have

$$\begin{aligned} X^2+Y^2+Z^2 &= x^2+y^2+z^2+2(\gamma z-\alpha y)\partial_x+2(\alpha x-\beta z)\partial_y+2(\beta y-\gamma x)\partial_z \\ &+ (\alpha^2+\gamma^2)\partial_x^2+(\alpha^2+\beta^2)\partial_y^2+(\beta^2+\gamma^2)\partial_z^2 \\ &- 2\beta\gamma\partial_x\partial_y-2\alpha\gamma\partial_y\partial_z-2\alpha\beta\partial_x\partial_z. \end{aligned} \tag{4.3}$$

We also introduce the geometric angular momentum

$$L=(L_x,L_y,L_z)=r\times\nabla, \qquad r=(r_x,r_y,r_z)=(x,y,z), \qquad \nabla=(\partial_x,\partial_y,\partial_z) \tag{4.4}$$

in terms of which the differential-operator representation may be written

$$\begin{aligned} V_{A_1}(X,Y,Z) &= V_{A_1}(x,y,z)+2\left(\alpha L_z+\beta L_x+\gamma L_y\right) \\ &+ (\alpha^2+\beta^2+\gamma^2)\nabla^2-(\alpha\partial_z+\beta\partial_x+\gamma\partial_y). \end{aligned} \tag{4.5}$$

The differential operators involved here are seen to satisfy the following commutator

$$\left[\alpha L_z+\beta L_x+\gamma L_y, (\alpha^2+\beta^2+\gamma^2)\nabla^2-(\alpha\partial_z+\beta\partial_x+\gamma\partial_y)\right]=0. \tag{4.6}$$

It is therefore natural to look for an orthonormal coordinate system (u,v,w) in terms of which $V_{A_1}(X,Y,Z)$ is independent of L_w. The two remaining coordinates are chosen from a 'symmetrical' point of view, as follows

$$\begin{aligned} x &= \frac{\gamma-\alpha}{N_2}u+\frac{\gamma(\gamma-\beta)+\alpha(\alpha-\beta)}{N_4}v+\frac{\beta}{N}w, \\ y &= \frac{\alpha-\beta}{N_2}u+\frac{\alpha(\alpha-\gamma)+\beta(\beta-\gamma)}{N_4}v+\frac{\gamma}{N}w, \\ z &= \frac{\beta-\gamma}{N_2}u+\frac{\beta(\beta-\alpha)+\gamma(\gamma-\alpha)}{N_4}v+\frac{\alpha}{N}w, \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} N_2^2 &= (\gamma-\alpha)^2+(\alpha-\beta)^2+(\beta-\gamma)^2, \\ N_4^2 &= (\gamma(\gamma-\beta)+\alpha(\alpha-\beta))^2+(\alpha(\alpha-\gamma)+\beta(\beta-\gamma))^2+(\beta(\beta-\alpha)+\gamma(\gamma-\alpha))^2, \\ N^2 &= \alpha^2+\beta^2+\gamma^2. \end{aligned} \tag{4.8}$$

These normalization constants are seen to be related according to

$$N_4^2=N_2^2N^2 \tag{4.9}$$

After some somewhat tedious computations one finds the following remarkable simplification

$$V_{A_1}(X,Y,Z) = u^2 + v^2 + w^2 + N^2(\partial_u^2 + \partial_v^2). \tag{4.10}$$

The simplicity of this expression is due to the change of coordinates (4.7), exploiting the symmetries of the original differential operator (4.3). In either form, the differential operator represents the NC elevation of the polynomial $V_{A_1}(x,y,z)$. It thus appears in the reduction of $\mathbb{C}^3_\Theta$ to the NC hypersurface defined by $V_{A_1}(X,Y,Z)=0$. Here we are interested in solving the differential equation (4.2) using (4.10).

The NC system may now be studied by representing the orthonormal coordinates (u,v,w) in cylindrical coordinates

$$u = \rho\cos(\theta), \qquad v = \rho\sin(\theta), \qquad w = w, \qquad \partial_u^2 + \partial_v^2 = \partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \frac{1}{\rho^2}\partial_\theta^2, \tag{4.11}$$

in which case the differential equation (4.2) reads

$$\left(\rho^2 + w^2 + N^2\left(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \frac{1}{\rho^2}\partial_\theta^2\right)\right)\Psi(\rho,\theta;w) = \epsilon\Psi(\rho,\theta;w). \tag{4.12}$$

Since this equation does not involve derivatives with respect to w, we consider the following simple separation of variables

$$\Psi(\rho,\theta;w) = R_w(\rho)\Upsilon(\theta) \tag{4.13}$$

with corresponding differential equations

$$\begin{aligned}\left(N^2\rho^2\frac{d^2}{d\rho^2} + N^2\rho\frac{d}{d\rho} + \rho^4 + (w^2-\epsilon)\rho^2 - \mu^2\right)R_{w,\mu}(\rho) &= 0,\\ \left(\frac{d^2}{d\theta^2} + \frac{\mu^2}{N^2}\right)\Upsilon_\mu(\theta) &= 0.\end{aligned} \tag{4.14}$$

The angular equation is the well-known differential equation for the harmonic oscillator. It has two linearly independent solutions:

$$\Upsilon_\mu^+(\theta) = e^{i\mu\theta/N}, \qquad \Upsilon_\mu^-(\theta) = e^{-i\mu\theta/N}. \tag{4.15}$$

After making the substitution

$$R(\rho) = \frac{1}{\rho}Q(i\rho^2/N) \tag{4.16}$$

for the radial function, we find the differential equation

$$\left(\frac{d^2}{d(\frac{i\rho^2}{N})^2} + \left(-\frac{1}{4} + \frac{i(\epsilon-w^2)/(4N)}{i\rho^2/N} + \frac{\frac{1}{4} - \left(\frac{\mu}{2N}\right)^2}{(i\rho^2/N)^2}\right)\right)Q(i\rho^2/N) = 0. \tag{4.17}$$

This is recognized as the Whittaker differential equation whose two linearly independent solutions may be represented by

$$M_{\frac{i(\epsilon-w^2)}{4N},\frac{\mu}{2N}}(i\rho^2/N) \quad \text{and} \quad M_{\frac{i(\epsilon-w^2)}{4N},-\frac{\mu}{2N}}(i\rho^2/N) \tag{4.18}$$

or in terms of Whittaker's function by

$$W_{\frac{i(\epsilon-w^2)}{4N},\frac{\mu}{2N}}(i\rho^2/N) \quad \text{and} \quad W_{-\frac{i(\epsilon-w^2)}{4N},\frac{\mu}{2N}}(-i\rho^2/N). \tag{4.19}$$

To see this, it is recalled [15] that the Whittaker differential equation is given by

$$\frac{d^2W(z)}{dz^2}+\left(-\frac{1}{4}+\frac{\lambda}{z}+\frac{\frac{1}{4}-\kappa^2}{z^2}\right)W(z)=0, \tag{4.20}$$

and that it has the two linearly independent solutions

$$\begin{aligned} M_{\lambda,\kappa}(z) &= z^{\kappa+\frac{1}{2}}e^{-z/2}\Phi(\kappa-\lambda+\frac{1}{2},2\kappa+1;z), \\ M_{\lambda,-\kappa}(z) &= z^{-\kappa+\frac{1}{2}}e^{-z/2}\Phi(-\kappa-\lambda+\frac{1}{2},-2\kappa+1;z). \end{aligned} \tag{4.21}$$

Here $\Phi(\nu,\tau;z)$ denotes the confluent hypergeometric function sometimes written ${}_1F_1(\nu;\tau;z)$. Whittaker's function provides solutions suitable for 2κ integer, and are defined by

$$W_{\lambda,\kappa}(z)=\frac{\Gamma(-2\kappa)}{\Gamma(\frac{1}{2}-\kappa-\lambda)}M_{\lambda,\kappa}(z)+\frac{\Gamma(2\kappa)}{\Gamma(\frac{1}{2}+\kappa-\lambda)}M_{\lambda,-\kappa}(z). \tag{4.22}$$

Two linearly independent solutions to (4.20) of this kind are given by $W_{\lambda,\kappa}(z)$ and $W_{-\lambda,\kappa}(-z)$.

Since the spectrum of solutions to the differential equation (4.2) is given in terms of the harmonic oscillator and solutions to the Whittaker differential equation, the involved parameters are not constrained by quantization conditions in the usual sense. Rather, the quantization conditions manifest themselves in the *form* of the spectrum which in this case is comprised of a combination of (4.15) and (4.19).

Now that we have the solution to (4.2) for all ϵ, it is natural to study the limit $\epsilon \to 0$. In the notation of (4.20), the only dependence on ϵ is through λ. The Whittaker differential equation and its solution are well defined for all λ, so we may conclude that the differential equation (4.17) is well defined for all ϵ. The original, classical A_1 geometry, on the other hand, is singular and corresponds to the aforementioned limit:

$$V_{A_1}(x,y,z)=\epsilon\to 0. \tag{4.23}$$

A merit of our quantization procedure is thus that the singularity of the classical geometry has been resolved. This NC elevation therefore offers an alternative to the more conventional resolutions.

Due to the interpretation that the differential operator (4.3) represents the NC elevation of the singular K3 surface $x^2+y^2+z^2=0$, we find it natural to attribute all the solutions to (4.2) to the 'moduli space' of the associated NC geometry. That is, the spectrum of wave functions solving (4.2), or more generally (2.4), for $\epsilon=0$ is interpreted as the moduli space of the NC geometry. Since the latter is represented by a partial differential equation, we see that

its possible boundary conditions correspond to constraints imposed on the moduli space. A detailed analysis of this link between boundary conditions and constraint equations is beyond the scope of the present work.

In the limit of zero non commutativity, i.e., $\alpha = \beta = \gamma = 0$, the change of coordinates (4.7) is singular. This is in accordance with the fact that the differential equation (4.2) merely reduces to its classical counterpart

$$(x^2 + y^2 + z^2)\Psi = \epsilon\Psi. \tag{4.24}$$

On the hypersurface (2.1) for $V = V_{A_1}$, every complex function solves (4.24). This means that the 'classical moduli space' may be identified with the set of holomorphic functions.

5 Discussion

We have developed a new and essentially non-geometric approach to NC Calabi-Yau manifolds, based on ideas from quantum mechanics. Our focus has been on the singular ADE geometries. The polynomial equations defining these classical ADE geometries are replaced by differential equations in which the original singularities are (presumably) absent. The moduli space associated to such an NC geometry is then interpreted as the spectrum of solutions to the corresponding wave equation. We have analyzed in detail the NC elevation of the A_1 geometry and found that it is described in part by the Whittaker differential equation. We intend to discuss elsewhere the extension of this explicit study to the other ADE geometries [6].

Our approach is adaptable to a broad variety of geometries whose NC elevations may then be represented by differential wave equations. The extension from complex to real variables is straightforward, as is the extension to other dimensions than two complex ones. We anticipate that the NC elevations of singular geometries in general will be non singular as in the case of A_1 discussed above. This will be addressed elsewhere [6] where we also intend to discuss the implementation of boundary conditions alluded to above.

In order to put the analogy between our construction and quantum mechanics to a 'physical test', one could examine the 'dual' descriptions of the NC elevations. That is, on one hand we have introduced the NC elevations as 'hypersurfaces' in an NC ambient space, while on the other hand we are representing them as differential operators resulting in some wave equations. In quantum mechanics, this corresponds to an operator description versus a description in terms of wave functions. It would therefore be of interest to try to extract information on an NC elevation based on both its dual descriptions. We believe that these complimentary approaches deserve to be studied further.

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